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A Study of Two Fractional Integral Problems Based on Jumarie Type of Riemann-Liouville Fractional Calculus

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus, we find the exact solutions of two fractional integrals. Integration by parts for fractional calculus and a new multiplication of fractional analytic functions play important roles in this article. In fact, our results are generalizations of classical calculus results.**

Keywords: **Jumarie type of R-L fractional calculus, fractional integrals, integration by parts for fractional calculus, new multiplication, fractional analytic functions.**

I. INTRODUCTION

Fractional calculus studies the so-called fractional integral and derivative of real or complex order and their applications. It originated in 1695, in a letter written by L'Hospital to Leibniz, some problems are proposed, such as "what does fractional derivative mean?" or "what is the 1⁄2 derivative of a function?" In the 18th and 19th centuries, many outstanding scientists focused their attention on this problem. For example, Euler, Laplace, Fourier, Abel, Liouville, Grünwald, Letnikov, Riemann, Laurent, or Heaviside. In the past few decades, fractional calculus has been applied to many fields, such as mechanics, economics, viscoelasticity, biology, control theory, and electrical engineering [1-15].

However, the definition of fractional derivative is not unique. Common definitions include Riemann-Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie's modified R-L fractional derivative [16-20]. Since Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified R-L fractional calculus, we obtain the exact solutions of the following two α-fractional integrals:

$$\left({}_{-\infty}I_{x}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes p}\otimes E_{\alpha}(qx^{\alpha})\right],$$

and

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_{t}^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(q-1)}\otimes[Ln_{\alpha}(t^{\alpha})]^{\otimes p}\right],$$

where $0<\alpha\leq 1$, $(-1)^{\alpha}$ exists and p,q are positive integers. Integration by parts for fractional calculus and a new multiplication of fractional analytic functions play important roles in this paper. In fact, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

Firstly, we introduce the fractional derivative used in this paper and its properties.

Definition 2.1 ([21])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ . \quad (1)$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\ , \quad (2)$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([22])**:** *If* α, β, x_0, C *are real numbers and* $\beta \geq \alpha > 0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \quad (3)$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \quad (4)$$

Next, we introduce the definition of fractional analytic function.

Definition 2.3 ([23])**:** If x, x_0, and a_k are real numbers for all k, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, i.e., $f_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\alpha+1)}(x-x_0)^{k\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, a new multiplication of fractional analytic functions is introduced below.

Definition 2.4 ([24])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. If $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \quad (5)$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\ . \quad (6)$$

Then we define

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \quad (7)$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha \sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \quad (8)$$

Definition 2.5 ([25])**:** If $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^\alpha\right)^{\otimes_\alpha n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\big(g_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\big(g_\alpha(x^\alpha)\big)^{\otimes_\alpha n}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\big(f_\alpha(x^\alpha)\big)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}. \tag{12}$$

Definition 2.6 ([26])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions. Then $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha n}=f_\alpha(x^\alpha)\otimes_\alpha\cdots\otimes_\alpha f_\alpha(x^\alpha)$ is called the nth power of $f_\alpha(x^\alpha)$. On the other hand, if $f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)=1$, then $g_\alpha(x^\alpha)$ is called the $\otimes_\alpha$ reciprocal of $f_\alpha(x^\alpha)$, and is denoted by $\big(f_\alpha(x^\alpha)\big)^{\otimes_\alpha -1}$.

Definition 2.7 ([27])**:** If $0<\alpha\leq 1$, and x is a real variable. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha n}. \tag{13}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$.

Theorem 2.8 (integration by parts for fractional calculus) ([28])**:** *Suppose that* $0<\alpha\leq 1$, a,b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha)\otimes_\alpha\left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right]=\left[f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\right]_{x=a}^{x=b}-\left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha)\otimes_\alpha\left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{14}$$

Notation 2.9: If r is any real number, p is any positive integer. Define $(r)_p=r(r-1)\cdots(r-p+1)$, and $(r)_0=1$.

III. MAIN RESULTS

In this section, we find the exact solutions of two fractional integrals.

Theorem 3.1: *If* $0<\alpha\leq 1$, $(-1)^\alpha$ *exists and* p,q *are positive integers, then the* α*-fractional integral*

$$\left({}_{-\infty}I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha p}\otimes_\alpha E_\alpha(qx^\alpha)\right]=E_\alpha(qx^\alpha)\otimes_\alpha\sum_{k=0}^{p}(-1)^k\frac{(p)_k}{q^{k+1}}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (p-k)}. \tag{15}$$

Proof : By integration by parts for fractional calculus,

$$\left({}_{-\infty}I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha p}\otimes E_\alpha(qx^\alpha)\right]$$

$$=\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha p}\otimes_\alpha\frac{1}{q}E_\alpha(qx^\alpha)\right]_{x=-\infty}^{x=x}-\left({}_{-\infty}I_x^\alpha\right)\left[\frac{1}{q}E_\alpha(qx^\alpha)\otimes_\alpha\left({}_{-\infty}D_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha p}\right]\right].$$

$$=\frac{1}{q}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha p}\otimes_\alpha E_\alpha(qx^\alpha)-\frac{p}{q}\left({}_{-\infty}I_x^\alpha\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (p-1)}\otimes_\alpha E_\alpha(qx^\alpha)\right]$$

$$=\cdots$$

$$=E_\alpha(qx^\alpha)\otimes_\alpha\sum_{k=0}^{p}(-1)^k\frac{(p)_k}{q^{k+1}}\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes_\alpha (p-k)}.$$

q.e.d.

Theorem 3.2: *Let* $0 < \alpha \leq 1$, $(-1)^{\alpha}$ *exists and* p, q *are positive integers, then the* α*-fractional integral*

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_t^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(q-1)}\otimes_{\alpha}[Ln_{\alpha}(t^{\alpha})]^{\otimes p}\right]=\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_{\alpha} q}\otimes_{\alpha}\sum_{k=0}^{p}(-1)^{k}\frac{(p)_k}{q^{k+1}}\left(Ln_{\alpha}(t^{\alpha})\right)^{\otimes_{\alpha}(p-k)}. \quad (16)$$

Proof: In Theorem 3.1, let $Ln_{\alpha}(t^{\alpha})=\frac{1}{\Gamma(\alpha+1)}x^{\alpha}$, then $\left(Ln_{\alpha}(t^{\alpha})\right)^{\otimes_{\alpha} p}=\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} p}$ and

$$\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_{\alpha} q}=E_{\alpha}\left(qLn_{\alpha}(t^{\alpha})\right)= E_{\alpha}(qx^{\alpha}). \quad (17)$$

And

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}D_t^{\alpha}\right)[Ln_{\alpha}(t^{\alpha})]=\left({}_{-\infty}D_x^{\alpha}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right]=1. \quad (18)$$

Thus, by Theorem 3.1

$$\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_t^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes(q-1)}\otimes[Ln_{\alpha}(t^{\alpha})]^{\otimes p}\right]$$

$$=\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}I_t^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes q}\otimes[Ln_{\alpha}(t^{\alpha})]^{\otimes p}\otimes_{\alpha}\left({}_{[\Gamma(\alpha+1)]^{\frac{1}{\alpha}}}D_t^{\alpha}\right)[Ln_{\alpha}(t^{\alpha})]\right]$$

$$=\left({}_{-\infty}I_x^{\alpha}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha} p}\otimes_{\alpha}E_{\alpha}(qx^{\alpha})\right]$$

$$=E_{\alpha}(qx^{\alpha})\otimes_{\alpha}\sum_{k=0}^{p}(-1)^{k}\frac{(p)_k}{q^{k+1}}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_{\alpha}(p-k)}$$

$$=\left(\frac{1}{\Gamma(\alpha+1)}t^{\alpha}\right)^{\otimes_{\alpha} q}\otimes_{\alpha}\sum_{k=0}^{p}(-1)^{k}\frac{(p)_k}{q^{k+1}}\left(Ln_{\alpha}(t^{\alpha})\right)^{\otimes_{\alpha}(p-k)}.$$

q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie's modified R-L fractional calculus, we find the exact solutions of two fractional integrals. Integration by parts for fractional calculus and a new multiplication of fractional analytic functions play important roles in this article. In fact, the major results we obtained are natural generalizations of the results in classical calculus. In the future, we will continue to use our methods to study the problems in applied mathematics and fractional differential equations.

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